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Scalar-valued measures of stress dispersion

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ABSTRACT

In situ stress is an important parameter in rock mechanics, but localised measurements often display significant variability; for meaningful analyses it is essential that such variability is appropriately quantified. Among many statistics, dispersion, which denotes how scattered or spread out a data group is, is an effective tool to quantify the amount of variability. However, dispersion measures are commonly only used for scalar and vector data, and it is not yet clear what robust scalar-valued measures of stress dispersion – i.e. measures that are faithful to the tensorial nature of stress – are available. Here, using stress tensors referred to a common Cartesian coordinate system, we consider several dispersion measures, namely, Euclidean dispersion (a tensor version of standard deviation), and the three widely used multivariate dispersions of total variation, generalised variance and effective variance, for scalar-valued quantification of stress variability and to improve the existing related work. We compare these measures, show how they are linked to the covariance matrix of tensor components, and derive their invariance with respect to change of coordinate system. Through the use of synthetic two-dimensional stress data we demonstrate that these measures can effectively characterise the dispersion of stress data. Further analysis of randomly generated three-dimensional stress data reveals that generalised variance and effective variance, which consider both variances of, and covariances between, tensor components, are more effective than Euclidean dispersion and total variation which ignore covariances. The transformational invariance of generalised variance and effective variance allows these measures to be applied in any convenient coordinate system.

1. Introduction

In situ stress is an important parameter for a wide range of endeavours in rock mechanics, including rock engineering design, hydraulic fracturing analysis, rock mass permeability and evaluation of earthquake potential.[1–5] The stress in rock often displays significant variability,[4,6–9] and as an example Fig. 1 shows the dramatic change in terms of both principal stress magnitude and orientation that can be observed in a small zone.[7] The stress variability may be influenced by various factors such as intrinsic variation caused by the inherent variability of discontinuities, anisotropy and heterogeneity of a fractured rock mass, as well as extrinsic errors related to stress acquisition methods. The acquisition error can be attributed to many aspects such as poor instrument installation, inaccurate estimation of mechanical parameters which are used in stress calculation, precision of the acquisition instruments, as well as the assumptions and constraints that have been made regarding the principal stresses in methods like hydraulic fracturing and borehole breakout analysis.[1,10,11] In addition, since stress may vary with respect to space (e.g. burial depth) and time, spatial variability and temporal variability also exist.[1] Therefore, the variability of stress is complicated in nature and robust statistical approaches are necessary and prerequisite to fully understand the complexity of stress variability. However, currently, such robust statistical approaches for stress variability characterisation are still lacking.

To alleviate the complexity and make the investigation of stress variability more realistic, assumptions have to be and have already been made,[1] and based on which, many examples of direct statistical processing of stress data can be found in the rock mechanics literature.[6,12–25] For example, it is common to assume that the analysed stresses were obtained within a space and time span that are sufficiently short such that their spatial and temporal variability can be ignored, and the measured stress data are deemed to be practically accurate. Based on these assumptions, several statistical approaches for stress data processing, such as mean stress calculation, statistical distribution model and confidence interval characterisation, have been developed.[12–19] However, in rock mechanics, assessment of stress variability is customarily undertaken by processing principal stress magnitude and orientation separately using scalar- or vector-related statistics (e.g. Fig. 2). This processing effectively decomposes the second order stress tensor into scalar (principal stress magnitudes) and vector (principal

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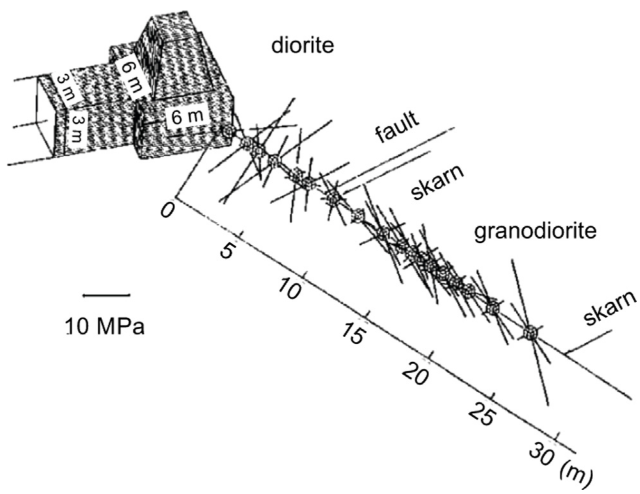
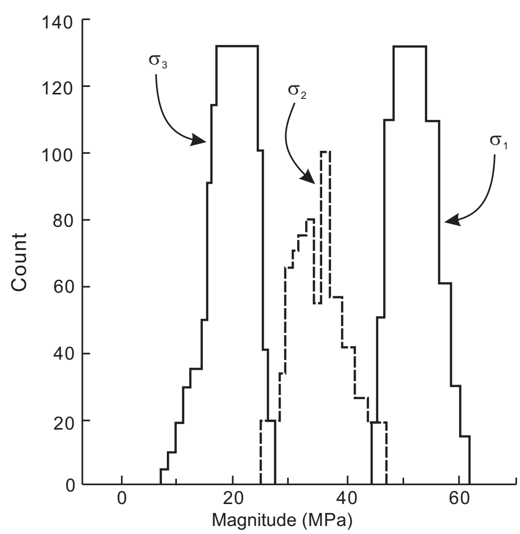
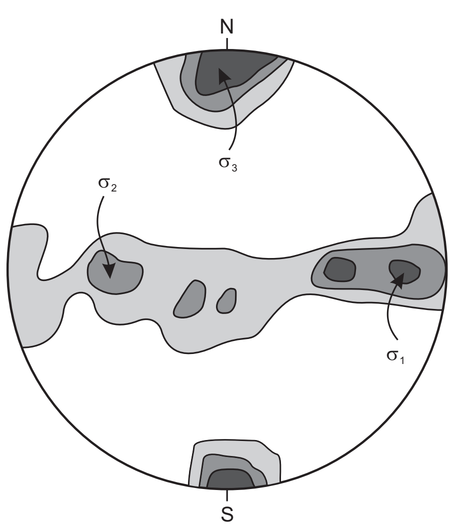


Fig. 1. Dramatic stress change in terms of both principal stress magnitude and orientation observed near a fault (from Obara & Sugawara[7]).



(a) Distribution of principal stress magnitudes



(b) Contouring of principal stress orientations

Fig. 2. Customary analyses of stress examine principal stress magnitude and orientation separately using classical statistics and directional statistics, respectively (after Brady & Brown[34]).

stress orientations) components, to which classical statistics[20] and directional statistics,[21] respectively, are applied. Examples of this approach are widespread in the literature.[6,22–34] All these customary methods not only violate the tensorial nature of stress, but also yield unreasonable results.[16,35–37]

Among many statistics, dispersion (also called scatter, denoting how spread out is a data group) is an effective tool to quantify variability, and it is commonly measured by standard deviation.[20(p.54)] However, standard deviation is only defined for scalar and vector data, and a robust approach to calculating the analogue of standard deviation for stress data is still not clear. This is mainly because of the tensorial nature of stress, which renders classical statistics inapplicable.[35,38] Particularly, for customary applications, when it comes to stress dispersion, one may intuitively calculate the dispersion of principal stress magnitude and orientation separately and hence obtain six dispersions. However, neither the six dispersions individually nor any combination of them gives a sense of the overall stress dispersion. A particular effect of this is that it is currently difficult to quantitatively evaluate overall stress variability, and impossible to quantitatively compare the variability of stress at different engineering sites. To overcome this shortfall and improve the existing related working in rock mechanics,[16,17,19,39,40] based on the above-mentioned assumptions, here we present and examine several dispersion measure approaches, and hence propose a scalar-valued stress dispersion measure for stress variability characterisation.

Rather than customary approaches that analyse principal stress magnitude and orientation separately, in order to remain faithful to the tensorial nature of stress, stress variability analysis should be conducted on the basis of tensor components obtained in a common Cartesian coordinate system. This has been advocated previously by many others.[16–19,39–43] Several researchers have followed this technique in stress dispersion related calculations.[17,19,39,40,42] For example, as dispersion is generally determined relative to the mean, it is necessary to first calculate the mean stress tensor as the mean of the stress tensors referred to a common frame. This approach first takes a group of n stress measurements in a global x-y-z Cartesian coordinate system, the ith stress tensor $\mathbf{S}_i$ of which is given by

$$\mathbf{S}_i = \begin{bmatrix} \sigma_{x_i} & \tau_{xy_i} & \tau_{xz_i} \\ & \sigma_{y_i} & \tau_{yz_i} \\ \text{symmetric} & & \sigma_{z_i} \end{bmatrix}, \tag{1}$$

where σ and τ are the normal and shear tensor components, respectively. The mean stress tensor is then[42]

$$\bar{\mathbf{S}}_{\mathrm{E}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i = \begin{bmatrix} \bar{\sigma}_x & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ & \bar{\sigma}_y & \bar{\tau}_{yx} \\ \text{symmetric} & & \bar{\sigma}_z \end{bmatrix}$$
$$= \frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n}\sigma_{x_i} & \sum_{i=1}^{n}\tau_{xy_i} & \sum_{i=1}^{n}\tau_{xz_i} \\ & \sum_{i=1}^{n}\sigma_{y_i} & \sum_{i=1}^{n}\tau_{yz_i} \\ \text{symmetric} & & \sum_{i=1}^{n}\sigma_{z_i} \end{bmatrix}, \tag{2}$$

where $\bar{\mathbf{S}}_{\mathrm{E}}$ denotes the Euclidean mean stress tensor,[44] and $\bar{\sigma}$ and $\bar{\tau}$ denote the corresponding mean tensor components. A number of reports exist in the literature in which this Euclidean mean has been used as a mean stress tensor.[16,17,19,39–41]

Based on Eq. (2), a so-called stress variance tensor may be calculated.[17,19,39,40] After obtaining the mean stress tensor, a new coordinate system (say, X-Y-Z) is established that coincides with the principal directions of the mean tensor $\bar{\mathbf{S}}_{\mathrm{E}}$, and all the original stress tensors transformed into this new coordinate system. Using the variance function, $\mathrm{var}(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, and recognising that $\bar{\tau}_{XY} = \bar{\tau}_{YZ} = \bar{\tau}_{ZX} = 0$, the variance tensor is then calculated as

$$\sigma_S^2 = \frac{1}{n-1}\begin{bmatrix} \sum_{i=1}^{n}(\sigma_{X_i}-\bar{\sigma}_X)^2 & \sum_{i=1}^{n}(\tau_{XY_i})^2 & \sum_{i=1}^{n}(\tau_{XZ_i})^2 \\ & \sum_{i=1}^{n}(\sigma_{Y_i}-\bar{\sigma}_Y)^2 & \sum_{i=1}^{n}(\tau_{YZ_i})^2 \\ \text{symmetric} & & \sum_{i=1}^{n}(\sigma_{Z_i}-\bar{\sigma}_Z)^2 \end{bmatrix}. \tag{3}$$

However, this only gives the dispersion of each tensor component, and an overall stress dispersion is not known.

As a development of this stress variance tensor approach, Dyke *et al.*[16] proposed a scalar-valued indication of the overall variability of a tensor group. After obtaining the stress tensors in a common coordinate system, say, *X-Y-Z*, each tensor component is processed individually to calculate its variance, and finally the sum of the variances of the nine tensor components, i.e. summation of all components in Eq. (3), is used as a measure of the overall stress dispersion. Although this approach indeed gives a scalar-valued measure of stress dispersion, its efficacy and transformational invariance have not been examined. As a result, to date there seems to have been no mathematically rigorous proposal and systematic examination in rock mechanics community for scalar-valued measures of stress dispersion.

In the present paper, in order to provide a robust approach to calculating the scalar-valued stress dispersion, well-developed knowledge from the statistics field is borrowed to extend what people have done earlier in rock mechanics.[16,17,19,39,40] To realise this, we first present the Euclidean dispersion, a matrix-based approach that offers a tensor version of standard deviation.[44] Then, since stress variability can be correctly and adequately represented by the variability of tensor components in a multivariate manner,[14,15,42,45,46] we continue and present three widely used multivariate measures of dispersion (which for brevity we refer to as "multivariate dispersion" hereafter) – namely, total variation,[47] generalised variance[48] and effective variance[49] – to calculate scalar-valued stress dispersions. The relationship between these stress dispersions is demonstrated and the derivations of their transformational invariance is given in an analytical manner. Finally, using synthetic and actual stress data, we show the applicability of these stress dispersion measures, and give recommendations for practical application.

The present work is part of our recent researches on solving a fundamental problem in rock mechanics – how stress data can be robustly processed in a statistical sense.[38,44–46,50,51] In the current work we mainly focus on proposing and examining scalar-valued measures of stress dispersion from a statistical and mathematical point of view, while considering the physical meaning of the stress tensor. Investigation of how various factors, such as measurement error, spatial and temporal variability, may contribute to the overall stress variability and how these factors can be distinguished in the calculation are beyond the scope of this work. The exclusion of these aspects means that the stress data used in the present work is assumed to be complete, practically accurate and obtained using the same approach within a short space and time span. Notations adopted here generally follow that bold uppercase, bold lowercase and normal lowercase letters denote matrix, vector and scalar, respectively, unless otherwise noted.

2. Scalar-valued measures of stress dispersion

To provide a robust approach for scalar-valued measures of stress dispersion, here we present the Euclidean dispersion, a scalar-valued dispersion we have defined previously,[44] together with several dispersion measures widely used in multivariate statistics that may be applicable to stress dispersion quantification, and examine the relationship between them.

2.1. Euclidean dispersion

Euclidean dispersion is a scalar-valued measure of the dispersion of a stress tensor group that considers each tensor as a single entity in Euclidean space. It is analogous to standard deviation, and can be interpreted as the square root of the second central moment of a collection of stress tensors.[44] Euclidean dispersion of stress tensors is defined in a similar fashion to standard deviation for scalars, and is based on the Euclidean distance between each tensor and the mean stress tensor.

For scalar datum x_i, the Euclidean distance between it and the mean $\bar{x}$ is

$$d(x_i, \bar{x}) = |x_i - \bar{x}|, \tag{4}$$

where $|\cdot|$ denotes the absolute value. The scalar standard deviation is[20(p.56)]

$$D(x_i;\, i \in 1{:}\, n) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} d^2(x_i, \bar{x})} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}|^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2}. \tag{5}$$

Here the n is used as the denominator, rather than $(n-1)$, since we want to emphasise that we "average" the results. For consistency, similar reasoning applies in the following where n appears in the denominator.

Continuing, the Euclidean dispersion of stress tensors is based on the Euclidean distance between tensors, which for the two stress tensors $\mathbf{S}_1$ and $\mathbf{S}_2$ is[52(p.117)]

$$d(\mathbf{S}_1, \mathbf{S}_2) = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} (\mathbf{S}_{ij|1} - \mathbf{S}_{ij|2})^2} = \|\mathbf{S}_1 - \mathbf{S}_2\|_{\mathrm{F}}, \tag{6}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm (also called the Euclidean norm). The Frobenius norm of a 3×3 stress tensor $\mathbf{S}$ is given by[53(p.72)]

$$\|\mathbf{S}\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{S}_{ij}^2} = \sqrt{\mathrm{tr}(\mathbf{S}\cdot\mathbf{S}^T)} = \sqrt{\mathrm{tr}(\mathbf{S}^2)}, \tag{7}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. Thus, the Euclidean dispersion of stress tensors is

$$D_{\mathrm{E}}(\mathbf{S}_i;\, i \in 1{:}\, n) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \|\mathbf{S}_i - \bar{\mathbf{S}}_{\mathrm{E}}\|_{\mathrm{F}}^2}, \tag{8}$$

where $\bar{\mathbf{S}}_{\mathrm{E}}$ is the Euclidean mean tensor (Eq. (2)), and clearly shows how spread out the stress data are with respect to their mean.[44]

2.2. Multivariate dispersions

Matrix-valued quantities play a pivotal role in many subjects such as solid mechanics, physics, earth science, medical imaging and economics,[54] and to characterise the variability of such matrix-valued quantities, matrix variate statistics – as a generalisation of multivariate statistics – has been developed.[54] We have previously examined the applicability of multivariate statistics and matrix variate statistics to stress tensor components.[45,55] Matrix variate statistics and multivariate statistics are often used interchangeably by statisticians,[54,56–58] and although matrix variate analysis of stress tensors and multivariate analysis of their components have been shown to be statistically equivalent,[45] difficulties associated with the application of matrix variate statistics to matrices such as stress tensors dictate that stress variability is better analysed in terms of tensor components in a multivariate manner.[45] Some analyses in this vein can be found in the literature.[14,15,42] As a result, it is reasonable to suggest that scalar-valued dispersion measures defined in multivariate statistics may be applied to the quantification of stress dispersion.

Three scalar-valued dispersion measures – total variation,[47] generalised variance[48] and effective variance[49] – are widely used in multivariate statistics. All of these are based on the covariance matrix of a collection of multivariate vectors. When applying multivariate dispersions to stress dispersion measure, two stress vectors need to be considered – one containing the complete tensor (i.e. either 4 or 9 components, associated with two- or three-dimensional stress states, respectively) and the other containing only the distinct tensor components (i.e. 3 or 6 components for two- or three-dimensional stress states, respectively). Based on the stress tensor **S** (Eq. (1)), these two vectors are

$$\mathbf{s}_\mathrm{c} = \mathrm{vec}(\mathbf{S}) = [\sigma_x \ \ \tau_{yx} \ \ \tau_{zx} \ \ \tau_{xy} \ \ \sigma_y \ \ \tau_{zy} \ \ \tau_{xz} \ \ \tau_{yz} \ \ \sigma_z]^T = [\sigma_x \ \ \tau_{xy} \ \ \tau_{xz} \ \ \tau_{yx} \ \ \sigma_y \ \ \tau_{yz} \ \ \tau_{zx} \ \ \tau_{zy} \ \ \sigma_z]^T \quad (9)$$

and

$$\mathbf{s}_\mathrm{d} = \mathrm{vech}(\mathbf{S}) = [\sigma_x \ \ \tau_{yx} \ \ \tau_{zx} \ \ \sigma_y \ \ \tau_{zy} \ \ \sigma_z]^T = [\sigma_x \ \ \tau_{xy} \ \ \tau_{xz} \ \ \sigma_y \ \ \tau_{yz} \ \ \sigma_z]^T. \quad (10)$$

Here, the subscripts "c" and "d" denote "complete" and "distinct", respectively, and $[\cdot]^T$ represents the matrix transpose. vec(·) is the vectorisation function that converts a tensor into a column vector by stacking all columns together,[59] and vech(·) is the half-vectorisation function that stacks only the lower triangular (i.e. on and below the diagonal) columns of a tensor into a column vector containing only the distinct components.[53(p.246)]

For vector $\mathbf{s}_\mathrm{c}$, containing all tensor components, the covariance matrix is

$$\mathbf{\Sigma} = \mathrm{cov}(\mathbf{s}_\mathrm{c}) = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{s}_{\mathrm{c}_i} - \bar{\mathbf{s}}_\mathrm{c})\cdot(\mathbf{s}_{\mathrm{c}_i} - \bar{\mathbf{s}}_\mathrm{c})^T, \quad (11)$$

where cov(·) denotes the covariance function[53(p.428)] and $\bar{\mathbf{s}}_\mathrm{c}$ is the mean vector such that

$$\bar{\mathbf{s}}_\mathrm{c} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{s}_{\mathrm{c}_i} = \mathrm{vec}(\bar{\mathbf{S}}_\mathrm{E}). \quad (12)$$

The total variation, which is the trace of the covariance matrix **Σ**, is thus

$$V_{\mathrm{t|c}} = \mathrm{tr}(\mathbf{\Sigma}). \quad (13)$$

Now, the duplicated rows and columns in **Σ** induced by the repeated matrix components in the stress vector $\mathbf{s}_\mathrm{c}$ means that **Σ** has a determinant of zero and hence is singular. The generalised variance and effective variance, which are related to the determinant of **Σ** (see Eqs. (17) and (18) below), are therefore not defined for the complete stress vector $\mathbf{s}_\mathrm{c}$.

For the stress vector $\mathbf{s}_\mathrm{d}$ containing only the distinct tensor components, its covariance matrix is

$$\mathbf{\Omega} = \mathrm{cov}(\mathbf{s}_\mathrm{d}) = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{s}_{\mathrm{d}_i} - \bar{\mathbf{s}}_\mathrm{d})\cdot(\mathbf{s}_{\mathrm{d}_i} - \bar{\mathbf{s}}_\mathrm{d})^T, \quad (14)$$

where $\bar{\mathbf{s}}_\mathrm{d}$ denotes the mean vector,which is calculated as

$$\bar{\mathbf{s}}_\mathrm{d} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{s}_{\mathrm{d}_i} = \mathrm{vech}(\bar{\mathbf{S}}_\mathrm{E}). \quad (15)$$

Based on the covariance matrix **Ω** given in Eq. (14), the total variation, generalised variance and effective variance are respectively given by

$$V_{\mathrm{t|d}} = \mathrm{tr}(\mathbf{\Omega}), \quad (16)$$

$$V_{\mathrm{g|d}} = |\mathbf{\Omega}| \quad (17)$$

and

$$V_{\mathrm{e|d}} = \sqrt[\frac{1}{2}p(p+1)]{|\mathbf{\Omega}|} = \sqrt[\frac{1}{2}p(p+1)]{V_{\mathrm{g|d}}}, \quad (18)$$

where |·| denotes the matrix determinant and p (p = 2, 3) is the dimension of the stress tensor.

2.3. Relations between the stress dispersion measures

The above definitions demonstrate that the stress dispersion measures depend on the variances of, and covariances between, the stress tensor components. Using Eq. (6) to expand the Euclidean dispersion in Eq. (8), we obtain

$$\begin{aligned} D_\mathrm{E}^2(\mathbf{S}_i;\, i \in 1{:}\, n) &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\|\mathbf{S}_i - \bar{\mathbf{S}}_\mathrm{E}\|_\mathrm{F}^2 \\ &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}((\sigma_{x_i} - \bar{\sigma}_x)^2 + 2(\tau_{xy_i} - \tau_{xy})^2 + 2(\tau_{xz_i} - \tau_{xz})^2 \\ &\quad + (\sigma_{y_i} - \sigma_y)^2 + 2(\tau_{yz_i} - \tau_{yz})^2 + (\sigma_{z_i} - \sigma_z)^2) \\ &= \mathrm{var}(\sigma_x) + 2\mathrm{var}(\tau_{xy}) + 2\mathrm{var}(\tau_{xz}) + \mathrm{var}(\sigma_y) + 2\mathrm{var}(\tau_{yz}) \\ &\quad + \mathrm{var}(\sigma_z) \end{aligned}, \quad (19)$$

demonstrating that the square of the Euclidean dispersion is the sum of the variances of all tensor components. Thus, we see that the measure offered by Dyke et al.[16] is in fact the Euclidean dispersion.

Since the leading diagonal of the covariance matrix **Σ** comprises the variances of all the tensor components, we therefore find that Euclidean dispersion and total variation of all tensor components are linked through

$$D_\mathrm{E}^2 = \mathrm{tr}(\mathbf{\Sigma}) = V_{\mathrm{t|c}}. \quad (20)$$

The total variation of distinct tensor components $\mathbf{s}_\mathrm{d}$ (Eq. (16)) is a summation of the variances of only the distinct tensor components, i.e.

$$V_{\mathrm{t|d}} = \mathrm{var}(\sigma_x) + \mathrm{var}(\tau_{xy}) + \mathrm{var}(\tau_{xz}) + \mathrm{var}(\sigma_y) + \mathrm{var}(\tau_{yz}) + \mathrm{var}(\sigma_z) = \mathrm{tr}(\mathbf{\Omega}), \quad (21)$$

and is thus different from the Euclidean dispersion.

The generalised variance and effective variance are only defined for stress vectors containing distinct tensor components. The square of Euclidean dispersion, total variation and effective variance all have the same units as the variance of the tensor components, i.e. (stress)^2 while the generalised variance has the awkwardly large unit of (stress)^{12}

Comparison between these scalar-valued stress dispersion measures also indicates that the covariances between tensor components (i.e. the off-diagonal components in the covariance matrix) are not considered in Euclidean dispersion and total variation; this is in contrast to the generalised and effective variances which consider not only the variances of, but also the covariances between, the tensor components. The effect of this on their capability for stress dispersion measure is discussed later.

3. Transformational invariance of the stress dispersion measures

To be meaningful, scalar-valued stress dispersion needs to be invariant with respect to coordinate transformation (i.e. independent of the coordinate system). Here we examine the transformational invariance of the stress dispersion measures discussed above.

3.1. Transformational invariance of Euclidean dispersion and total variation

As the Euclidean dispersion is based on the Euclidean distance between tensors, we examine Eq. (6) to consider the case when the stress tensor **S** is subject to transformation represented by the transformation matrix **R**, i.e.

$$\mathbf{S}' = \mathbf{R}\cdot\mathbf{S}\cdot\mathbf{R}^T. \quad (22)$$

Here, **S′** denotes the stress tensor in a new coordinate system corresponding to the transformation matrix **R**. For the Euclidean distance between two transformed tensors we therefore have

$$\begin{aligned} d^2(\mathbf{S}_1', \mathbf{S}_2') &= \|\mathbf{S}_1' - \mathbf{S}_2'\|_\mathrm{F}^2 = \|\mathbf{R}\mathbf{S}_1\mathbf{R}^T - \mathbf{R}\mathbf{S}_2\mathbf{R}^T\|_\mathrm{F}^2 = \|\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)\mathbf{R}^T\|_\mathrm{F}^2 \\ &= \mathrm{tr}(\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)\mathbf{R}^T\cdot\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)\mathbf{R}^T) = \mathrm{tr}(\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)^2\mathbf{R}^T) \\ &= \mathrm{tr}((\mathbf{S}_1 - \mathbf{S}_2)^2) = \|\mathbf{S}_1 - \mathbf{S}_2\|_\mathrm{F}^2 = d^2(\mathbf{S}_1, \mathbf{S}_2). \end{aligned} \quad (23)$$

This confirms the transformational invariance of Euclidean distance, and thus that of the Euclidean dispersion and the total variation of complete tensor components. However, this invariance does not apply to the total variation of the distinct tensor components, as is shown later in the analysis of actual stress data.

3.2. Transformational invariance of generalised variance and effective variance

As shown above, both generalised and effective variances are related to the determinant of the covariance matrix of distinct tensor components, $|\boldsymbol{\Omega}|$. Therefore, provided transformational invariance of $|\boldsymbol{\Omega}|$ is satisfied, i.e. $|\boldsymbol{\Omega}'| = |\boldsymbol{\Omega}|$, transformational invariance of the generalised variance and effective variance are guaranteed. This invariance has previously been derived analytically by Gao & Harrison.[45] This derivation makes use of a number of specialised matrix manipulations and identities. To begin, it is necessary to write Eq. (14) in terms of $\mathbf{s}_c$ instead of $\mathbf{s}_d$. This is achieved by using the relation

$$\mathbf{s}_d = \mathbf{B}_p^T \cdot \mathbf{s}_c, \tag{24}$$

where the transition matrix $\mathbf{B}_p$, of dimensions $p^2 \times \frac{1}{2}p(p+1)$, eliminates the duplicated tensor components of $\mathbf{s}_c$.[53(p.246),54(p.11)] The matrices $\mathbf{B}_p$ corresponding to two- and three-dimensional stress tensors are respectively

$$\mathbf{B}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{B}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{25}$$

By this means, after lengthy matrix manipulation it is found that the determinant of the covariance matrix of distinct components $|\boldsymbol{\Omega}|$ and the determinant of the covariance matrix of all components $|\boldsymbol{\Sigma}|$ are related via[45]

$$|\boldsymbol{\Omega}| = |\mathbf{B}_p^T \boldsymbol{\Sigma} \mathbf{B}_p| = 2^{-\frac{1}{2}p(p-1)} \cdot |\mathbf{B}_p^+ \boldsymbol{\Sigma} \mathbf{B}_p|, \tag{26}$$

where $\mathbf{B}_p^+$ is the Moore-Penrose pseudoinverse of $\mathbf{B}_p$,[60(pp.36–38)] i.e.

$$\mathbf{B}_p^+ \mathbf{B}_p = \mathbf{I}_{\frac{1}{2}p(p+1)}, \tag{27}$$

and $\mathbf{I}$ is the identity matrix with dimensions of $\frac{1}{2}p(p+1)$.

Additionally, the determinant of the transformed covariance matrix may be written as[45]

$$|\boldsymbol{\Omega}'| = |\mathbf{B}_p^T \cdot (\mathbf{R} \otimes \mathbf{R}) \cdot \boldsymbol{\Sigma} \cdot (\mathbf{R}^T \otimes \mathbf{R}^T) \cdot \mathbf{B}_p|, \tag{28}$$

where $\otimes$ denotes the Kronecker product. Again, after lengthy manipulation using a range of specialised matrix identities,[45] Eq. (28) reduces to

$$|\boldsymbol{\Omega}'| = 2^{-\frac{1}{2}p(p-1)} \cdot |\mathbf{B}_p^+ \boldsymbol{\Sigma} \mathbf{B}_p|. \tag{29}$$

Comparison of Eqs. (26) and (29) shows their right hand sides to be identical, and thus

$$|\boldsymbol{\Omega}| = |\boldsymbol{\Omega}'|. \tag{30}$$

This confirms the transformational invariance of the determinant of covariance matrix $\boldsymbol{\Omega}$, with the result that the generalised variance and effective variance of distinct tensor components are independent of the coordinate system. This allows these dispersion measures to be used in any convenient coordinate system, and opens the way for them to be used in a practical setting.

4. Application, comparison and discussion

To give a detailed application and examination of the proposed scalar-valued stress dispersion measures, here we use both two- and three-dimensional stress data to investigate their applicability and efficacy, as well as to confirm their transformational invariance. Recommendations are given for use of the measures in practice.

4.1. Two-dimensional stress data application

We start with the analysis of two-dimensional stress data. Two groups of synthetic stress data, each comprising 20 tensors, are obtained by introducing random perturbations to a base stress tensor. The perturbations are drawn from uniform distributions, the ranges of which, together with the base stress tensor, are shown in Table 1. The two generated synthetic stress data groups are shown in Table 2. As the perturbations of group B are larger than those of group A, we might expect group B to show larger dispersion than group A. This is supported by the Mohr's circles of the two stress groups shown in Fig. 3, where data group A appears more concentrated than group B. Dispersions of the two stress data groups calculated using the measures introduced above are tabulated in Table 3, and this shows all dispersions of data group B to have larger values than those of data group A. This suggests that for this case all the stress dispersion measures capture the dispersion of the two data groups.

4.2. Three-dimensional stress data application

For three-dimensional stress data application, 17 actual *in situ* stress data obtained at a depth of around 417 m as part of the *in situ* stress measurements made at the Atomic Energy of Canada Limited (AECL)'s Underground Research Laboratory (URL) in south-eastern Manitoba, Canada[6] are used to verify the transformational invariance and performance of the stress dispersion measures. Geomechanical research

Table 1
Base stress tensor and the perturbation ranges applied to the base stress tensor components for Data group A and B.

Tensor components	Base tensor components (MPa)	Perturbation ranges	
		Group A	Group B
σ_x	30.00	[−2,2]	[−4,4]
τ_{xy}	5.00	[−1,1]	[−2,2]
σ_y	15.00	[−2,2]	[−4,4]

Table 2
Two synthetic stress data groups.

Data group A (MPa)			Data group B (MPa)		
σ_x	τ_{xy}	σ_y	σ_x	τ_{xy}	σ_y
28.71	5.79	15.52	27.42	6.58	16.05
29.44	4.14	13.36	28.88	3.29	11.72
28.23	4.48	13.32	26.45	3.97	11.65
30.09	4.11	16.11	30.18	3.22	17.22
29.34	4.88	16.62	28.69	4.77	18.24
28.70	4.03	15.14	27.41	3.05	15.27
28.84	5.79	13.44	27.67	6.59	11.87
31.62	4.39	16.30	33.24	3.79	17.61
30.70	4.19	14.35	31.40	3.37	13.70
29.87	4.61	14.18	29.75	4.23	13.35
31.65	4.91	15.99	33.30	4.82	16.97
28.42	4.20	13.04	26.83	3.41	11.08
30.98	5.99	13.19	31.96	6.98	11.39
30.95	4.66	15.67	31.89	4.33	16.34
30.25	4.59	15.41	30.49	4.19	15.83
28.74	4.12	15.10	27.47	3.25	15.21
30.39	4.60	15.92	30.78	4.19	16.84
29.20	4.09	15.83	28.40	3.19	16.66
28.54	5.01	16.13	27.07	5.02	17.25
28.85	5.52	14.15	27.70	6.05	13.30

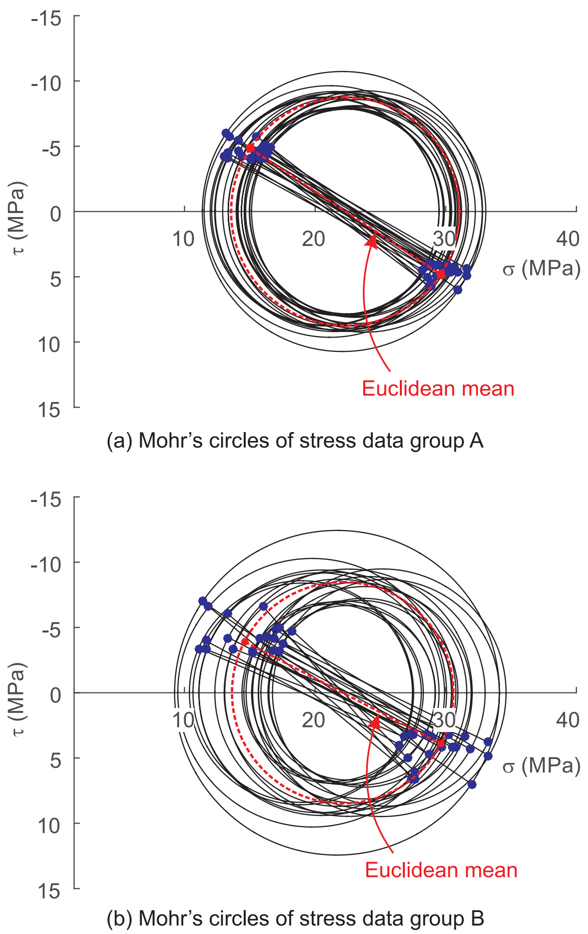


Fig. 3. Mohr's circles of the two synthetic stress data groups.

Table 3
Scalar-valued stress dispersion measures of the two synthetic stress data groups.

Dispersions	Data group A	Data group B
D_E^2[a]	3.22 MPa^2	12.89 MPa^2
$V_{t\|c}$[b]	3.22 MPa^2	12.89 MPa^2
$V_{t\|d}$	2.85 MPa^2	11.41 MPa^2
$V_{g\|d}$	0.49 MPa^6	31.38 MPa^6
$V_{e\|d}$	0.79 MPa^2	3.15 MPa^2

[a] Calculated using Eq. (8).
[b] Calculated using Eq. (13).

was conducted at the AECL's URL during around 1982–2004 to assess the feasibility of nuclear fuel waste disposal deep in a plutonic rock mass.[6,61] These 17 stress data are part of the 99 *in situ* stress measurements presented in Martin,[6] which were conducted at the AECL's URL using the modified South African Council of Scientific and Industrial Research (CSIR) triaxial strain cell.[62] Here, these 17 actual stress data are assumed to satisfy the above-mentioned assumptions and only used for the purpose of demonstrating the applicability and efficacy of the proposed approach from mathematical and statistical point of view. The 17 actual stress data, transformed into the common coordinate system of x East, y North and z vertically upwards, are presented in Table 4.

4.2.1. Transformational invariance of the proposed stress dispersion measures

The mean tensor and covariance matrix of the 17 stress tensors in Table 4 are calculated using Eqs. (2), (11) and (14). Referred to an x-y-z coordinate system, these are respectively

$$\bar{\mathbf{s}}_d = [34.84 \quad -0.30 \quad -3.61 \quad 40.36 \quad 1.67 \quad 15.35]^T \text{ MPa} \tag{31}$$

and

$$\mathbf{\Omega} = \begin{bmatrix} 67.59 & 34.96 & 1.74 & -42.09 & 0.11 & 7.01 \\ & 63.61 & 0.72 & -40.24 & -1.75 & 7.86 \\ & & 1.43 & -2.92 & -0.63 & -0.59 \\ & & & 58.29 & 0.38 & -6.85 \\ & & & & 3.33 & 0.63 \\ \text{symmetric} & & & & & 4.13 \end{bmatrix} \text{MPa}^2. \tag{32}$$

Table 4
In situ stress tensor components in the x-y-z coordinate system (data from Martin[6]).

Depth (m)	Stress tensor components (MPa)					
	σ_x	τ_{xy}	τ_{xz}	σ_y	τ_{yz}	σ_z
416.55	43.25	4.67	−3.44	32.67	−0.34	15.35
416.57	41.20	6.59	−3.32	31.30	0.46	17.69
416.60	42.92	8.80	−3.97	35.83	2.83	14.57
416.62	45.11	5.42	−4.44	31.59	2.29	18.34
416.68	42.57	4.36	−1.93	28.27	0.85	15.13
416.69	53.78	5.26	−2.26	31.51	3.62	17.61
416.70	26.05	−7.48	−2.57	38.40	1.74	12.35
416.71	28.85	−12.01	−5.65	45.40	6.71	16.29
416.73	30.96	−9.73	−3.86	42.67	0.45	14.56
416.77	23.88	−9.88	−3.70	51.36	1.09	15.19
416.79	34.97	−14.97	−4.51	57.51	1.80	11.74
416.81	27.89	−10.89	−1.60	44.53	−0.24	14.22
417.17	33.78	6.06	−2.19	46.27	0.19	14.59
417.17	33.09	6.35	−5.77	45.00	0.10	18.15
417.17	26.07	4.60	−3.30	42.37	3.14	12.69
417.17	28.18	4.70	−3.89	40.82	3.72	18.25
417.17	29.73	3.00	−4.92	40.55	−0.08	14.22

To test the transformational invariance of the proposed stress dispersion measures, another coordinate system (X-Y-Z), which is aligned with the direction of the principal components of the mean stress tensor, is used. The principal stress directions are the eigenvectors of the Euclidean mean stress tensor, and are found to be

$$\mathbf{R}^T = \begin{bmatrix} 0.1037 & -0.9792 & -0.1743 \\ -0.9913 & -0.1160 & 0.0615 \\ -0.0805 & 0.1664 & -0.9828 \end{bmatrix}, \tag{33}$$

where the three column vectors correspond to the directions of σ_1, σ_2 and σ_3, respectively, referred to the x-y-z frame.

After transforming the 17 stress data into the X-Y-Z system using Eq. (22), the mean tensor and covariance matrix are found to be

$$\bar{\mathbf{s}}'_d = [40.52 \quad 0 \quad 0 \quad 35.42 \quad 0 \quad 14.61]^T \text{ MPa} \tag{34}$$

and

$$\mathbf{\Omega}' = \begin{bmatrix} 74.53 & -38.37 & -11.65 & -56.12 & -7.95 & -9.17 \\ & 45.74 & 9.07 & 32.75 & 0.97 & 5.46 \\ & & 5.79 & 10.52 & 0.43 & 2.01 \\ & & & 78.81 & 10.49 & 9.86 \\ & & & & 3.35 & 0.62 \\ \text{symmetric} & & & & & 3.67 \end{bmatrix} \text{MPa}^2, \tag{35}$$

respectively.

The non-zero off-diagonal elements in both Eqs. (32) and (35) indicate that the correlations between the various distinct stress tensor components are not zero, i.e. the stress tensor components are not independent.[20(p.73)] This prompts us to suggest using the term "six distinct components" rather than "six independent components" to refer to stress tensor components, in order to be statistically correct[63(p.56)] as well as to avoid misinterpretations.[38,45]

All the proposed stress dispersion measures can be calculated using the covariance matrices of the tensor components, i.e. $\mathbf{\Sigma}$ and $\mathbf{\Omega}$. The

Table 5
Stress dispersion measures of actual stress data in *x-y-z* and *X-Y-Z* coordinate systems.

Dispersions	Coordinate systems	
	x-y-z	*X-Y-Z*
D_E^2, $V_{t\|c}$	266.77 MPa^2	266.77 MPa^2
$V_{t\|d}$	198.39 MPa^2	211.89 MPa^2
$V_{g\|d}$	6.57×10^5 MPa^{12}	6.57×10^5 MPa^{12}
$V_{e\|d}$	9.33 MPa^2	9.33 MPa^2

measures obtained using the covariance matrices associated with the *x-y-z* and *X-Y-Z* coordinate systems are shown in Table 5, and demonstrates that Euclidean dispersion, total variation of complete tensor components, generalised variance and effective variance display transformational invariance. Total variation of distinct tensor components does not display this invariance, and thus is not recommended for practical use. Additional calculations using different coordinate systems, but not presented here for the sake of brevity, support this finding. As a result, it appears that Euclidean dispersion, total variation of complete tensor components, generalised variance and effective variance, can quantify stress dispersion in any convenient coordinate system and are therefore suitable for general use.

4.2.2. Efficacy of the stress dispersion measures

The generally applicable stress dispersion measures fall into two categories: those – such as Euclidean dispersion and total variation – that are based on the trace of the covariance matrix of complete tensor components, tr($\mathbf{\Sigma}$), and those – such as generalised variance and effective variance – that are based on the determinant of the covariance matrix of the distinct tensor components, $|\mathbf{\Omega}|$. As noted above, the former only consider the variances of tensor components, while the latter also consider the covariances between tensor components. One approach to examining the effect of this difference on the efficacy of these measures is to generate appropriate random stress tensors, evaluate the various dispersion measures and then compare these to the distributions of the principal stress magnitudes and orientations.

Random stress tensors can be generated in a multivariate manner using the mean tensor and the covariance matrix of the distinct tensor components as inputs, an approach we have described previously.[38] This earlier work used the AECL data described above to generate two groups of random tensors using the mean vector given in Eq. (31) but different covariance matrices. These generated data are used here to test the various dispersion measures.

The first group of random tensors are generated using the covariance matrix of the 17 actual stress data referred to the *x-y-z* coordinate system, i.e. Eq. (32), while the second group uses this same matrix but with zero off-diagonal elements, i.e.

$$\mathbf{\Omega}_{\rm II} = \begin{bmatrix} 67.59 & 0 & 0 & 0 & 0 & 0 \\ & 63.61 & 0 & 0 & 0 & 0 \\ & & 1.43 & 0 & 0 & 0 \\ & & & 58.29 & 0 & 0 \\ & & & & 3.33 & 0 \\ \text{symmetric} & & & & & 4.13 \end{bmatrix} \text{MPa}^2. \tag{36}$$

Thus, both stress groups have the same Euclidean dispersion and total variation (266.77 MPa^2), but different generalised and effective variance: for the first group these are 6.57×10^5 MPa^{12} and 9.33 MPa^2, respectively, while for the second group they are 4.94×10^6 MPa^{12} and 13.05 MPa^2. These figures indicate that the dispersion of the second group is larger than that of the first, and thus we would expect the

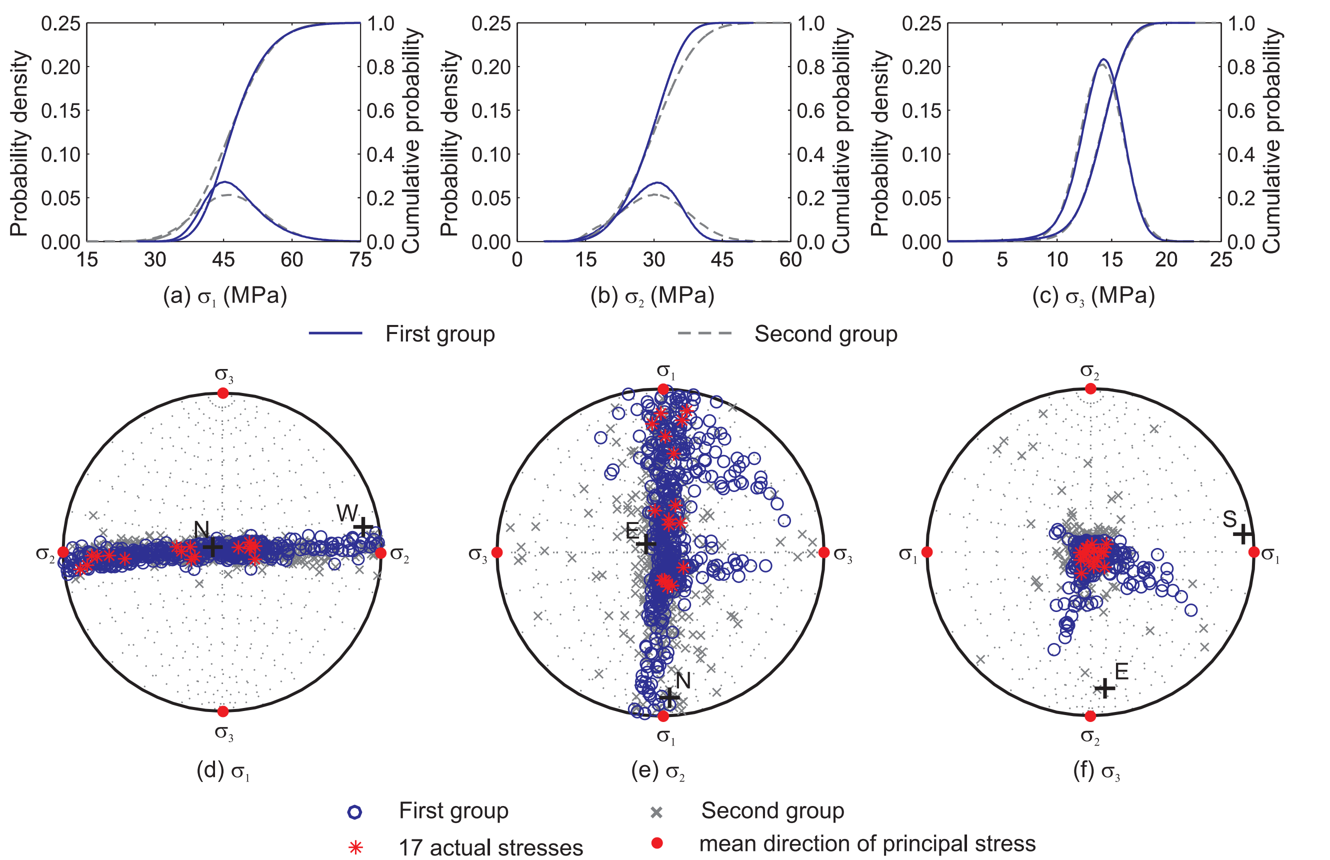


Fig. 4. Distributions of principal stress magnitudes and hemispherical projections of principal stress orientations of two groups of random tensors (for clarity the projections show a random selection of only 500 generated tensors, and have been rotated to place the mean at the centre of the projection and the other two principal stress directions at the N-S and E-W positions) (after Gao & Harrison[38]).

Table 6
Angular differences between the mean principal directions of the 500-tensor subset and the population for the two groups of generated stress data (from Gao & Harrison[38]).

	σ_1 (°)	σ_2 (°)	σ_3 (°)
Group 1: covariance matrix of Eq. (32)	1.2	1.2	0.3
Group 2: covariance matrix of Eq. (36)	1.2	1.1	0.4

second group to display larger variation in terms of principal stress magnitudes and orientations than does the first.

The variability of principal stress magnitudes and orientations was investigated in our earlier work[38] by generating 5×10^6 random tensors for each group as an accurate representation of the population. From these random tensors the probability density and cumulative distributions of the generated principal stresses were determined, and equal angle lower hemispherical projections of the principal directions of a randomly drawn subset of 500 tensors plotted (Fig. 4a-c and Fig. 4d-f, respectively). This smaller number of vectors has been used for clarity; note that the angular differences between the mean principal directions of the 500 tensors and the population are practically insignificant, as shown in Table 6.

As Fig. 4 shows, the distributions of the magnitudes of σ_1 and σ_2, and the orientations of σ_2 and σ_3 of the first group (i.e. non-zero covariance) appear more concentrated than those associated with the second group (i.e. zero covariance), with the distributions of the magnitude of σ_3 and orientation of σ_1 being approximately the same for both groups. Ignoring the covariance (i.e. group 2) therefore appears to give greater dispersion than if covariance is correctly included. By extension, dispersion measures that ignore covariance will not correctly account for the dispersion of the data.

This demonstrates that, by ignoring the covariances between tensor components, Euclidean dispersion and total variation do not capture the full stress dispersion. Consequently, we suggest that generalised variance and effective variance are the more effective in assessing stress dispersion. Furthermore, since effective variance has a relatively small magnitude and also a more appreciable unit (the same as the variance of tensor components, i.e. stress squared), we recommend this for practical use.

Effectively, this analysis compares two groups of *in situ* stress data that have the same Euclidean mean but different variability. As the effective variance is different for the two groups, we suggest this measure will be efficacious for making inter-site comparisons of *in situ* stress variability.

4.2.3. Influence of tensor component sequence on stress dispersion measure

The sequence of the distinct tensor components used above is that generated by the vech(·) function in Eq. (10). If the sequence is changed to

$$\mathbf{s}_{\mathrm{d}} = [\tau_{xy} \quad \tau_{xz} \quad \tau_{yz} \quad \sigma_x \quad \sigma_y \quad \sigma_z]^T, \tag{37}$$

and the 17 actual stress data shown in Table 4 reanalysed, the resulting covariance matrix is

$$\mathbf{\Omega}_{\mathrm{III}} = \begin{bmatrix} 63.61 & 0.72 & -1.75 & 34.96 & -40.24 & 7.86 \\ & 1.43 & -0.63 & 1.74 & -2.92 & -0.59 \\ & & 3.33 & 0.11 & 0.38 & 0.63 \\ & & & 67.59 & -42.09 & 7.01 \\ & & & & 58.29 & -6.85 \\ \text{symmetric} & & & & & 4.13 \end{bmatrix} \mathrm{MPa}^2. \tag{38}$$

Comparing Eqs. (32)–(38) shows that the elements of the covariance matrices are identical, although in a different sequence. The effective variance calculated based on this new covariance matrix is 9.33 MPa^2, which is the same that obtained with the original sequence (Table 5). Thus, the sequence of stress tensor components is seen to have no effect on the effective variance, and hence stress dispersion quantification can be conducted using any convenient sequence of distinct tensor components.

5. Conclusions and further comments

By improving on the existing related working in rock mechanics, we have presented and examined several stress dispersion quantification approaches – Euclidean dispersion, total variation, generalised variance and effective variance – to provide scalar-valued measures of the overall stress variability based on the stress tensors referred to a common Cartesian coordinate system.

Comparison between these stress dispersion measures demonstrates that the Euclidean dispersion and total variation of complete tensor components are identical and equal to the sum of the variances of all tensor components, while the generalised variance and effective variance are only defined for distinct tensor components and both are related to the determinant of their covariance matrix.

We have examined analytically the transformational invariance of the stress dispersion measures. Euclidean dispersion, total variation of complete tensor components, generalised variance and effective variance of distinct tensor components are seen to be independent of the coordinate system, but total variation of distinct tensor components is coordinate system dependent and is therefore not suitable for practical use. Analysis of actual stress data confirms these results.

Testing of the efficacy of the stress dispersion measures using randomly generated stress data demonstrates that for different stress groups which have the same Euclidean dispersion and total variation, they may have different overall dispersions. This is because the Euclidean dispersion and total variation do not account for the covariances between tensor components, and thus the Euclidean dispersion and total variation do not fully capture the overall stress dispersion. This is in contrast to the generalised variance and effective variance, both of which consider the variances of and covariances between tensor components and hence are more effective measures of stress dispersion. Finally, since effective variance has a relatively small magnitude and familiar units of stress squared, it is more convenient and thus recommended for engineering application.

We suggest that the recommended scalar-valued stress dispersion measure – effective variance – may not only assist in stress variability characterisation, but also facilitate the quantitative comparison of the variability of stress under different scenarios, such as those obtained from different engineering sites.

However, as mentioned earlier, the effective variance is applicable in situations where the stress data are complete, practically accurate and obtained using the same approach within a short space and time span. While since stress measurement methods involve perturbation of a particular volume of rock, and the volume involved for different methods may differ by several orders of magnitude. For example, the earthquake focal mechanism method may involve a rock volume of 10^9 m^3. Hydraulic fracturing method involves somewhat smaller rock volumes (0.5–50 m^3), but still larger than those associated with overcoring techniques (10^{-3}–10^{-2} m^3).[1] How the stress data obtained by different approaches can be processed together in a statistical manner is a significant challenge in stress variability analyses. In addition, the spatial and temporal variability may be an intrinsic feature of stress. Currently, the proposed approach is incapable of capturing spatial and temporal variability of stress. Geostatistics, which is a branch of statistics focusing on spatial or spatiotemporal datasets, may provide a potential solution to this problem.[64] All these problems are worth future attentions of the rock mechanics community. Nevertheless, the proposed scalar-valued measures of stress dispersion, together with other tensor-based stress variability characterisation approaches we have developed earlier,[38,44–46,50,51] may provide a quantitative and precise assistance to solve these problems.

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